

Comment on “The complete Schwarzschild interior and exterior
solution in the harmonic coordinate system”

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In a recent paper Liu [1] considered the complete Schwarzschild interior and exterior solution in harmonic coordinates. There he argued about the necessity to keep the integration constant C_1 in R_{ex} , in contrast with previous treatments (Refs. 1-5 and Ref. 8 of Ref. 1). The purpose of this comment is to show that the above conclusion cannot be traced from the matching conditions between the vacuum exterior and the uniform density interior perfect fluid, as claimed in [1]. The reason for this is that the last condition in Eqs. (7) of Ref. 1, namely $R'_{in}(a) = R'_{ex}(a)$ is *not* required by the junction conditions at $r = a$, as will be shown in what follows.

The junction of two space-times along a timelike hypersurface Σ can be done applying the Darmois-Israel matching procedure [2, 3], which requires the continuity across the junction of both the first and second fundamental forms (induced metric and extrinsic curvature) *of the junction hypersurface*.

In the standard coordinates, the metric (1) of Ref. 1 induces the 3-metric given by the line element

$$ds_\Sigma^2 = E(r)dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

on the junction hypersurface $r = a$, which has the normal $n = 1/\sqrt{G} \partial/\partial r$ and the nonvanishing extrinsic curvature components

$$K_{00} = -\frac{E'}{2\sqrt{G}}, \quad K_{22} = \frac{r}{\sqrt{G}}, \quad K_{33} = \frac{r}{\sqrt{G}} \sin^2\theta. \quad (2)$$

The continuity of both of (1) and (2) implies that the metric function G is continuous and E is \mathcal{C}^1 across the junction. It is not to be expected that starting from the same metrics written in other coordinate systems the conditions on E and G will be weakened. However, constraints on the new function R introduced by the coordinate transformation (2)-(5) of Ref. 1 will also emerge.

In the harmonic coordinate system the normal vector to Σ has the components $n^\mu = (\ln R)'/\sqrt{G} (0, X_1, X_2, X_3)$. The metric (6) of Ref. 1 (with a missing square on the last bracket corrected) induces the first fundamental form

$$ds_\Sigma^2 = E(r)dt^2 - \frac{r^2}{R^2(r)}d\mathbf{X}^2. \quad (3)$$

This is still expressed in terms of the four space-time coordinates. When written in terms of the coordinates t, θ and ϕ , intrinsic to Σ , the continuity of the induced metric again implies the continuity of the metric function E alone. The extrinsic curvature tensor in the new

coordinate system is found either by direct computation or by transforming its components (2) from standard to harmonic coordinates. The nonvanishing components are K_{00} given in (2) and

$$K_{ii} = \frac{(X_j^2 + X_k^2)r}{R^4\sqrt{G}} , \quad K_{ij} = -\frac{X_i X_j r}{R^4\sqrt{G}} , \quad (4)$$

where $i \neq j \neq k$ take the values 1,2,3. The junction condition on the extrinsic curvature at arbitrary radius implies that G, R and E' should be continuous. Altogether we find that G and R are \mathcal{C}^0 and E is \mathcal{C}^1 . Thus the last relation in (7) of Ref. 1 does not hold.

The condition

$$E'_{in} = E'_{ex} , \quad (5)$$

though not listed among the continuity conditions (7) of Ref. 1, was fulfilled when imposing that the pressure vanishes on the junction. Indeed, Eq. (5) is a substitute for the requirement that the radial pressures on the two sides of Σ are equal, which was demonstrated for generic spherically symmetric static space-times in an other context [4].

Our criticism does not affect the main result of Ref. 1, which is the solution R_{in} of the second order differential equation (22) of Ref. 1. The arguments about the integration constants however should be reviewed. Requiring only the continuity of R and nothing more, one of the constants C_1 and C_2 can be freely specified, in particular $C_1 = 0$ can be chosen, in accordance with Refs. 1-5 and Ref. 8 of Ref. 1. Of course, the continuity of R' across the junction of the interior and exterior Schwarzschild solutions can be imposed as an additional requirement for other purposes (e.g. for having a smooth function $R(r)$ as in Ref. 1), but it is not a consequence of the junction conditions.

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